Labor demand decreases in the wage

Proof by revealed profitability

This proof that labor demand slopes downward is adapted from Hal Varian's textbook *Intermediate Microeconomics* (5'th ed.), chapter 19.10.

Consider a firm producing one output using two inputs, labor and capital. Given the output price p, the wage w, and the capital cost r, it chooses optimal amounts of output (q), labor (E) and capital (K), to maximize profits (π) given the technology constraints.

The firm's profits are

$$\pi = pq - wE - rK$$

Suppose that the firm chooses the quantities (q_1, E_1, K_1) under the set of prices (p_1, w_1, r_1), and (q_2, E_2, K_2) under the prices (p_2, w_2, r_2).

If the technology is unchanged between the two situations, both sets of quantities are feasible under both sets of prices. If the firm maximizes profits, the set of quantities chosen under each set of prices generate more profits than the other quantities would have done. Therefore the following inequalities hold:

$$p_1 q_1 - w_1 E_1 - r_1 K_1 \geq p_1 q_2 - w_1 E_2 - r_1 K_2 \tag{1}$$

$$p_2 q_2 - w_2 E_2 - r_2 K_2 \geq p_2 q_1 - w_2 E_1 - r_2 K_1 \tag{2}$$

Negating both sides of the inequality (1) and swaping the two sides so that the greater-than-sign is retained, yields:

$$-p_1 q_2 + w_1 E_2 + r_1 K_2 \geq -p_1 q_1 + w_1 E_1 + r_1 K_1 \tag{3}$$

Adding the inequalities (2) and (3) yields:

$$(p_2 - p_1) q_2 - (w_2 - w_1) E_2 - (r_2 - r_1) K_2 \geq (p_2 - p_1) q_1 - (w_2 - w_1) E_1 - (r_2 - r_1) K_1 \tag{4}$$

and subtracting the right hand side from the left hand side:

$$(p_2 - p_1)(q_2 - q_1) - (w_2 - w_1)(E_2 - E_1) - (r_2 - r_1)(K_2 - K_1) \geq 0 \tag{5}$$

Introducing the following notation for increases in prices and quantities:

$\Delta p = p_2 - p_1,\ \Delta w = w_2 - w_1,\ \Delta r = r_2 - r_1,\ \Delta q = q_2 - q_1,\ \Delta E = E_2 - E_1,$
$\Delta K = K_2 - K_1,$

the inequality (5) is written:

$$\Delta p \Delta q - \Delta w \Delta E - \Delta r \Delta K \geq 0 \tag{6}$$

If only the wage (w) changes and other prices remain unchanged ($\Delta p = \Delta r = 0$), the inequality (6) reduces to:

$$-\Delta w \Delta E \geq 0 \tag{7}$$

or

$$\Delta w \Delta E \leq 0 \tag{8}$$

The last inequality states that w and E move in oposite directions, i.e. employment responds negatively to wage increases.